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The reciprocal sums of even and odd terms in the Fibonacci sequence

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Abstract

In this paper, we investigate the reciprocal sums of even and odd terms in the Fibonacci sequence, and we obtain four interesting families of identities which give the partial finite sums of the even-indexed (resp., odd-indexed) reciprocal Fibonacci numbers and the even-indexed (resp., odd-indexed) squared reciprocal Fibonacci numbers.

MSC: 11B39

Keywords: Fibonacci sequence; even and odd term; reciprocal sum

1 Introduction

The *Fibonacci sequence* is defined by the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2,$$

where F_n is called the nth *Fibonacci number* with $F_0 = 0$ and $F_1 = 1$. There exists a simple and non-obvious formula for the Fibonacci numbers,

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The Fibonacci sequence plays an important role in the theory and applications of mathematics, and its various properties have been investigated by many authors; see [1–5].

In recent years, there has been an increasing interest in studying the reciprocal sums of the Fibonacci numbers. For example, Elsner *et al.* [6–9] investigated the algebraic relations for reciprocal sums of the Fibonacci numbers. In [10], the partial infinite sums of the reciprocal Fibonacci numbers were studied by Ohtsuka and Nakamura. They established the following results, where $\lfloor\cdot\rfloor$ denotes the floor function.

Theorem 1.1 *For all* $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k}\right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even}; \\ F_{n-2} - 1, & \text{if } n \text{ is odd}. \end{cases} \tag{1.1}$$

Theorem 1.2 *For each $n \geq 1$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \text{ is even;} \\ F_n F_{n-1}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

Wu and Zhang [11, 12] generalized these identities to the Fibonacci polynomials and Lucas polynomials, and they considered the subseries of infinite sums derived from the reciprocals of the Fibonacci polynomials and Lucas polynomials.

Recently, Wu and Wang [13] studied the partial finite sum of the reciprocal Fibonacci numbers and deduced the following main result.

Theorem 1.3 *For all $n \geq 4$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}. \tag{1.3}$$

Inspired by Wu and Wang's work, Wang and Wen [14] strengthened Theorem 1.1 and 1.2 to the finite sum case.

Theorem 1.4 *If $m \geq 3$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd.} \end{cases} \tag{1.4}$$

Theorem 1.5 *For all $m \geq 2$ and $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \text{ is even;} \\ F_n F_{n-1}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.5}$$

Applying elementary methods, we investigate the partial finite sums of the even-indexed and odd-indexed reciprocal Fibonacci numbers in this paper, and obtain four interesting families of identities. In Section 2, we consider the reciprocal sums of even and odd terms in the Fibonacci sequence. In Section 3, we present the finite sums of the even-indexed and odd-indexed squared reciprocal Fibonacci numbers.

2 Main results I: the reciprocal sums

We first present several well-known results on Fibonacci numbers, which will be used throughout the article. The detailed proofs can be found in [5].

Lemma 2.1 *Let $n \geq 1$, we have*

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1} \tag{2.1}$$

and

$$F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1} \tag{2.2}$$

if a and b are positive integers.

As a consequence of (2.2), we have the following result.

Corollary 2.2 *If $n \geq 1$, then*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \tag{2.3}$$

$$F_{2n+1} = F_{n-1}F_{n+1} + F_nF_{n+2}, \tag{2.4}$$

$$F_{2n+1} = F_{n+1}F_{n+2} - F_{n-1}F_n. \tag{2.5}$$

The following is an interesting identity concerning the Fibonacci numbers.

Lemma 2.3 *Assume that a and b are two integers with $a \geq b \geq 0$. If $n > a$, then*

$$F_{n+a}F_{n-a-1} - F_{n+b}F_{n-b-1} = (-1)^{n-a}F_{a+b+1}F_{a-b}. \tag{2.6}$$

Proof We proceed by induction on n. It is clearly true for $n = a + 1$. Assuming the result holds for any integer $n > a$, we show that the same is true for $n + 1$.

Applying (2.2) repeatedly and by the induction hypothesis, we get

$$\begin{aligned}
F_{(n+1)+a}F_{(n+1)-a-1} - F_{(n+1)+b}F_{(n+1)-b-1} &= (F_{n+1+a}F_{n-a} + F_{n+a}F_{n-a-1}) \\
&\quad - F_{n+1+b}F_{n-b} - F_{n+a}F_{n-a-1} \\
&= F_{2n} - F_{n+1+b}F_{n-b} - F_{n+a}F_{n-a-1} \\
&= F_{n+b}F_{n-b-1} - F_{n+a}F_{n-a-1} \\
&= -(F_{n+a}F_{n-a-1} - F_{n+b}F_{n-b-1}) \\
&= (-1)^{n+1-a}F_{a+b+1}F_{a-b},
\end{aligned}$$

which completes the induction on n. □

Remark Recently, Akyiğit *et al.* [15, 16] defined the split Fibonacci quaternion, the split Lucas quaternion and the split generalized Fibonacci quaternion, and they obtained some similar identities to those above for these quaternions.

Before presenting our main results, we establish an inequality.

Proposition 2.4 *If $n \geq 3$, then*

$$\frac{1}{F_{4n+1}} > \sum_{k=n}^{2n} \frac{1}{F_{2k-1}F_{2k}F_{2k+1}}. \tag{2.7}$$

Proof A direct calculation shows that it is true for $n = 3$. Thus, we assume that $n \geq 4$ in the rest of the proof.

Setting $a = 2$ and $b = 0$, and replacing n by $2n$ in (2.6) yields

$$F_{2n+2}F_{2n-3} = F_{2n}F_{2n-1} + 2. \tag{2.8}$$

From (2.5), we know that

$$F_{4n+1} = F_{2n+1}F_{2n+2} - F_{2n-1}F_{2n}. \tag{2.9}$$

Applying (2.8), (2.9), and the fact $F_{2n-3} \geq 2$ and $F_{2n-1}F_{2n} > F_{2n+1}$ if $n \geq 3$, we obtain

$$\begin{aligned} F_{2n-3}F_{4n+1} &= F_{2n-3}(F_{2n+1}F_{2n+2} - F_{2n-1}F_{2n}) \\ &= F_{2n-3}F_{2n+1}F_{2n+2} - F_{2n-3}F_{2n-1}F_{2n} \\ &= (F_{2n-1}F_{2n} + 2)F_{2n+1} - F_{2n-3}F_{2n-1}F_{2n} \\ &= F_{2n-1}F_{2n}F_{2n+1} + 2F_{2n+1} - F_{2n-3}F_{2n-1}F_{2n} \\ &= F_{2n-1}F_{2n}F_{2n+1} - (F_{2n-3}F_{2n-1}F_{2n} - 2F_{2n+1}) \\ &< F_{2n-1}F_{2n}F_{2n+1}, \end{aligned}$$

which is equivalent to

$$\frac{F_{4n+1}}{F_{2n-1}F_{2n}F_{2n+1}} < \frac{1}{F_{2n-3}}.$$

Now we have

$$\begin{aligned} \frac{1}{F_{4n+1}} - \sum_{k=n}^{2n} \frac{1}{F_{2k-1}F_{2k}F_{2k+1}} &= \frac{1}{F_{4n+1}} - \frac{1}{F_{4n+1}} \sum_{k=n}^{2n} \frac{F_{4n+1}}{F_{2k-1}F_{2k}F_{2k+1}} \\ &> \frac{1}{F_{4n+1}} - \frac{1}{F_{4n+1}} \sum_{k=n}^{2n} \frac{1}{F_{2n-3}} \\ &= \frac{1}{F_{4n+1}} \left(\frac{F_{2n-3} - n - 1}{F_{2n-3}} \right). \end{aligned}$$

It is not hard to see that for $n \geq 4$, $F_{2n-3} \geq n + 1$, which completes the proof. □

Now we introduce our main results on the reciprocal sums of Fibonacci numbers.

Theorem 2.5 *For all* $n \geq 3$, *we have*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = F_{2n-1}. \tag{2.10}$$

Proof By elementary manipulations and (2.1), we derive that, for $k \geq 1$,

$$\begin{aligned} \frac{1}{F_{2k-1}+1} - \frac{1}{F_{2k}} - \frac{1}{F_{2k+1}+1} &= \frac{F_{2k}(F_{2k+1} - F_{2k-1}) - (F_{2k-1}+1)(F_{2k+1}+1)}{F_{2k}(F_{2k-1}+1)(F_{2k+1}+1)} \\ &= \frac{F_{2k}^2 - F_{2k-1}F_{2k+1} - F_{2k-1} - F_{2k+1} - 1}{F_{2k}(F_{2k-1}+1)(F_{2k+1}+1)} \\ &= \frac{(-1)^{2k-1} - F_{2k-1} - F_{2k+1} - 1}{F_{2k}(F_{2k-1}+1)(F_{2k+1}+1)}. \end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{k=n}^{2n}\frac{1}{F_{2k}} &= \frac{1}{F_{2n-1}+1}-\frac{1}{F_{4n+1}+1}+\sum_{k=n}^{2n}\left(\frac{1}{F_{2k}(F_{2k-1}+1)}+\frac{1}{F_{2k}(F_{2k+1}+1)}\right)\\ &> \frac{1}{F_{2n-1}+1}-\frac{1}{F_{4n+1}+1}+\frac{1}{F_{2n}(F_{2n-1}+1)}.\end{aligned}$$

It follows from (2.4) that

$$F_{4n+1} > F_{2n}F_{2n+2} > F_{2n}(F_{2n-1}+1),$$

which implies that

$$\sum_{k=n}^{2n}\frac{1}{F_{2k}} > \frac{1}{F_{2n-1}+1}. \tag{2.11}$$

Invoking (2.1) again, we can readily deduce that

$$\frac{1}{F_{2k-1}}-\frac{1}{F_{2k}}-\frac{1}{F_{2k+1}}=\frac{-1}{F_{2k-1}F_{2k}F_{2k+1}}, \tag{2.12}$$

from which we obtain

$$\sum_{k=n}^{2n}\frac{1}{F_{2k}}=\frac{1}{F_{2n-1}}-\frac{1}{F_{4n+1}}+\sum_{k=n}^{2n}\frac{1}{F_{2k-1}F_{2k}F_{2k+1}}.$$

Because of (2.7), we get, if $n \geq 3$,

$$\sum_{k=n}^{2n}\frac{1}{F_{2k}} < \frac{1}{F_{2n-1}}. \tag{2.13}$$

Combining (2.11) and (2.13), we have

$$\frac{1}{F_{2n-1}+1} < \sum_{k=n}^{2n}\frac{1}{F_{2k}} < \frac{1}{F_{2n-1}},$$

which yields the desired identity. □

Theorem 2.6 *If* $m \geq 3$ *and* $n \geq 1$, *we have*

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{1}{F_{2k}}\right)^{-1}\right\rfloor = F_{2n-1}-1. \tag{2.14}$$

Proof It is obviously true for $n = 1$. Now we assume that $n \geq 2$.

By some calculations and (2.1), we obtain, for $k \geq 2$,

$$\frac{1}{F_{2k-1}-1}-\frac{1}{F_{2k}}-\frac{1}{F_{2k+1}-1}=\frac{F_{2k-1}+F_{2k+1}-2}{(F_{2k-1}-1)F_{2k}(F_{2k+1}-1)} > 0, \tag{2.15}$$

from which we have

$$\sum_{k=n}^{mn} \frac{1}{F_{2k}} < \frac{1}{F_{2n-1}-1} - \frac{1}{F_{2mn+1}-1} < \frac{1}{F_{2n-1}-1}. \tag{2.16}$$

On the other hand, it follows from (2.12) that

$$\begin{aligned} \sum_{k=n}^{mn} \frac{1}{F_{2k}} &= \frac{1}{F_{2n-1}} - \frac{1}{F_{2mn+1}} + \sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}F_{2k+1}} \\ &> \frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1}F_{2n}F_{2n+1}} - \frac{1}{F_{2mn+1}}. \end{aligned}$$

We claim that if $n \geq 1$ and $m \geq 3$,

$$F_{2n-1}F_{2n}F_{2n+1} < F_{2mn+1}.$$

Replacing a by $a-1$ in (2.2), we arrive at

$$F_{a-1}F_b + F_aF_{b+1} = F_{a+b},$$

which implies that

$$F_{a+b} \geq F_aF_{b+1} \geq F_aF_b. \tag{2.17}$$

Thus, $F_{2n-1}F_{2n}F_{2n+1} \leq F_{6n} < F_{6n+1} \leq F_{2mn+1}$, which means

$$\sum_{k=n}^{mn} \frac{1}{F_{2k}} > \frac{1}{F_{2n-1}}. \tag{2.18}$$

Combining (2.16) and (2.18) yields

$$\frac{1}{F_{2n-1}} < \sum_{k=n}^{mn} \frac{1}{F_{2k}} < \frac{1}{F_{2n-1}-1},$$

from which the desired result follows immediately. □

Corollary 2.7 *For all $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = F_{2n-1} - 1. \tag{2.19}$$

Proof By using (2.15) repeatedly, we have

$$\begin{aligned} \frac{1}{F_{2n-1}-1} &> \frac{1}{F_{2n}} + \frac{1}{F_{2n+1}-1} \\ &> \frac{1}{F_{2n}} + \frac{1}{F_{2n+2}} + \frac{1}{F_{2n+3}-1} \\ &> \frac{1}{F_{2n}} + \frac{1}{F_{2n+2}} + \frac{1}{F_{2n+4}} + \cdots. \end{aligned}$$

Thus, we obtain

$$\sum_{k=n}^{\infty} \frac{1}{F_{2k}} < \frac{1}{F_{2n-1} - 1}.$$

Applying the same argument to (2.12) yields

$$\sum_{k=n}^{\infty} \frac{1}{F_{2k}} > \frac{1}{F_{2n-1}}.$$

Hence we have

$$\frac{1}{F_{2n-1}} < \sum_{k=n}^{\infty} \frac{1}{F_{2k}} < \frac{1}{F_{2n-1} - 1},$$

which completes the proof. □

Remark Identity (2.19) can be regarded as the limit of (2.14) as $m \to \infty$.

Theorem 2.8 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-2}. \tag{2.20}$$

Proof It is clearly true for $n = 1$, hence we suppose that $n \geq 2$ in the following.

Invoking (2.1), we derive that for $k \geq 2$,

$$\frac{1}{F_{2k-2}} - \frac{1}{F_{2k-1}} - \frac{1}{F_{2k}} = \frac{1}{F_{2k-2}F_{2k-1}F_{2k}} > 0,$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} < \frac{1}{F_{2n-2}} - \frac{1}{F_{2mn}} < \frac{1}{F_{2n-2}}. \tag{2.21}$$

It follows from (2.17) that

$$F_{4n+1} > F_{2n+1}F_{2n} > (F_{2n-2} + 1)F_{2n-1},$$

based on which we conclude that, when $n > 1$,

$$\frac{F_{2n} + F_{2n-2}}{(F_{2n-2} + 1)F_{2n-1}(F_{2n} + 1)} > \frac{1}{(F_{2n-2} + 1)F_{2n-1}} > \frac{1}{F_{4n+1}} \geq \frac{1}{F_{2mn+1}}.$$

Employing (2.1) again, we can readily obtain

$$\frac{1}{F_{2k-2} + 1} - \frac{1}{F_{2k-1}} - \frac{1}{F_{2k} + 1} = \frac{-F_{2k} - F_{2k-2}}{(F_{2k-2} + 1)F_{2k-1}(F_{2k} + 1)},$$

from which we arrive at

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_{2k-1}} &= \frac{1}{F_{2n-2}+1}-\frac{1}{F_{2mn}+1}+\sum_{k=n}^{mn}\frac{F_{2k}+F_{2k-2}}{(F_{2k-2}+1)F_{2k-1}(F_{2k}+1)}\\ &> \frac{1}{F_{2n-2}+1}-\frac{1}{F_{2mn}+1}+\frac{F_{2n}+F_{2n-2}}{(F_{2n-2}+1)F_{2n-1}(F_{2n}+1)}\\ &> \frac{1}{F_{2n-2}+1}.\end{aligned}$$

Combining the above inequality with (2.21), we have

$$\frac{1}{F_{2n-2}+1}<\sum_{k=n}^{mn}\frac{1}{F_{2k-1}}<\frac{1}{F_{2n-2}},$$

which yields the desired result. □

As m approaches infinity, Theorem 2.8 becomes the following.

Corollary 2.9 *If $n \geq 1$, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_{2k-1}}\right)^{-1}\right\rfloor = F_{2n-2}. \tag{2.22}$$

3 Main results II: the reciprocal square sums

We first introduce several preliminary results on the square of the Fibonacci numbers.

Lemma 3.1 *For all $n \geq 2$, we have*

$$F_{n-1}^2F_{n+1}^2 - F_{n-2}^2F_{n+2}^2 = (-1)^n\cdot 4\cdot F_n^2. \tag{3.1}$$

Proof It follows from

$$\begin{aligned}F_{n-2}F_{n+2} &= (F_n - F_{n-1})(F_n + F_{n+1})\\ &= F_n^2 + F_nF_{n+1} - F_{n-1}F_n - F_{n-1}F_{n+1}\\ &= 2F_n^2 - F_{n-1}F_{n+1}\end{aligned}$$

that

$$\begin{aligned}F_{n-1}^2F_{n+1}^2 - F_{n-2}^2F_{n+2}^2 &= (F_{n-1}F_{n+1}+F_{n-2}F_{n+2})(F_{n-1}F_{n+1}-F_{n-2}F_{n+2})\\ &= 2F_n^2\big(2F_{n-1}F_{n+1}-2F_n^2\big)\\ &= (-1)^n\cdot 4\cdot F_n^2,\end{aligned}$$

where the last equality follows from (2.1). □

Lemma 3.2 *If $n \geq 2$, then*

$$F_{2n-3}F_{2n+1} - F_{2n-1}^2 = 1. \tag{3.2}$$

Proof It is straightforward to check that

$$\begin{aligned} F_{2n-3}F_{2n+1} - F_{2n-1}^2 &= (F_{2n-1} - F_{2n-2})(F_{2n-1} + F_{2n}) - F_{2n-1}^2 \\ &= F_{2n-1}^2 + F_{2n-1}F_{2n} - F_{2n-2}F_{2n-1} - F_{2n-2}F_{2n} - F_{2n-1}^2 \\ &= F_{2n-1}F_{2n} - F_{2n-2}F_{2n-1} - F_{2n-2}F_{2n} \\ &= F_{2n-1}F_{2n} - F_{2n-2}F_{2n+1} \\ &= 1, \end{aligned}$$

where the last equality follows from (2.6). □

Lemma 3.3 *For each $n \geq 2$, we have*

$$F_{2n+1}^2 - F_{2n-3}^2 > 3F_{2n-1}^2. \tag{3.3}$$

Proof A direct calculation shows that

$$\begin{aligned} F_{2n+1}^2 - F_{2n-3}^2 &= (F_{2n+1} + F_{2n-3})(F_{2n+1} - F_{2n-3}) \\ &= (2F_{2n-1} + F_{2n-2} + F_{2n-3})(F_{2n-1} + F_{2n} - F_{2n-3}) \\ &> 3F_{2n-1}F_{2n-1} \\ &= 3F_{2n-1}^2. \end{aligned}$$

The proof is complete. □

Remark In fact, applying the equalities (ii) and (iv) of Proposition 2.2 of [17], we can easily obtain

$$F_{2n+1}^2 - F_{2n-3}^2 = (F_{2n+1} + F_{2n-3})(F_{2n+1} - F_{2n-3}) = 3F_{2n-1} \cdot L_{2n-1},$$

where L_n means the nth Lucas number. Then (3.3) follows immediately from the fact $L_n > F_n$ for $n \geq 2$.

Now we are ready to present the reciprocal square sums of the Fibonacci numbers.

Theorem 3.4 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}^2} \right)^{-1} \right\rfloor = F_{4n-2} - 1. \tag{3.4}$$

Proof It is clearly true for $n = 1$, so we assume that $n \geq 2$ in the rest of the proof.

For $k \geq 2$, we have

$$\begin{aligned} \frac{1}{F_{4k-2} - 1} - \frac{1}{F_{2k}^2} - \frac{1}{F_{4k+2} - 1} &= \frac{F_{2k}^2(F_{4k+2} - F_{4k-2}) - (F_{4k-2} - 1)(F_{4k+2} - 1)}{(F_{4k-2} - 1)F_{2k}^2(F_{4k+2} - 1)} \\ &> \frac{F_{2k}^2(F_{4k+2} - F_{4k-2}) - F_{4k-2}F_{4k+2} + F_{4k+2}}{(F_{4k-2} - 1)F_{2k}^2(F_{4k+2} - 1)}. \end{aligned}$$

It follows from (2.3) that

$$F_{2k}^2 - F_{2k-2}^2 = F_{4k-2}, \tag{3.5}$$

$$F_{2k+2}^2 - F_{2k}^2 = F_{4k+2}. \tag{3.6}$$

As a consequence of (3.1), we see

$$F_{2k-1}^2 F_{2k+1}^2 - F_{2k-2}^2 F_{2k+2}^2 = 4F_{2k}^2. \tag{3.7}$$

Applying (2.1), (3.5), (3.6), and (3.7), we derive that

$$\begin{aligned}
&F_{2k}^2(F_{4k+2} - F_{4k-2}) - F_{4k-2}F_{4k+2} + F_{4k+2} \\
&\quad = F_{2k}^2\left(F_{2k+2}^2 - 2F_{2k}^2 + F_{2k-2}^2\right) - \left(F_{2k}^2 - F_{2k-2}^2\right)\left(F_{2k+2}^2 - F_{2k}^2\right) + F_{2k+2}^2 - F_{2k}^2 \\
&\quad = -F_{2k}^4 + F_{2k-2}^2 F_{2k+2}^2 + F_{2k+2}^2 - F_{2k}^2 \\
&\quad = -F_{2k}^2\left(F_{2k}^2 + 1\right) + F_{2k-2}^2 F_{2k+2}^2 + F_{2k+2}^2 \\
&\quad = -(F_{2k-1}F_{2k+1} - 1)F_{2k-1}F_{2k+1} + F_{2k-2}^2 F_{2k+2}^2 + F_{2k+2}^2 \\
&\quad = -F_{2k-1}^2 F_{2k+1}^2 + F_{2k-1}F_{2k+1} + F_{2k-2}^2 F_{2k+2}^2 + F_{2k+2}^2 \\
&\quad = F_{2k+2}^2 - F_{2k-1}^2 F_{2k+1}^2 + F_{2k-2}^2 F_{2k+2}^2 + F_{2k-1}F_{2k+1} \\
&\quad = F_{2k+2}^2 - 4F_{2k}^2 + F_{2k-1}F_{2k+1} \\
&\quad = (F_{2k+2} - 2F_{2k})(F_{2k+2} + 2F_{2k}) + F_{2k-1}F_{2k+1} \\
&\quad > 0,
\end{aligned}$$

which implies that

$$\frac{1}{F_{4k-2} - 1} - \frac{1}{F_{2k}^2} - \frac{1}{F_{4k+2} - 1} > 0.$$

Thus, we have

$$\sum_{k=n}^{mn} \frac{1}{F_{2k}^2} < \frac{1}{F_{4n-2} - 1} - \frac{1}{F_{4mn+2} - 1} < \frac{1}{F_{4n-2} - 1}. \tag{3.8}$$

Employing the same argument as above, we obtain, for $k \geq 2$,

$$\frac{1}{F_{4k-2}} - \frac{1}{F_{2k}^2} - \frac{1}{F_{4k+2}} = -\frac{3F_{2k}^2 - F_{2k-1}F_{2k+1}}{F_{4k-2}F_{2k}^2 F_{4k+2}}.$$

For each $k \geq 2$, we have

$$\begin{aligned}
3F_{2k}^2 - F_{2k-1}F_{2k+1} &= 3F_{2k}^2 - F_{2k-1}(F_{2k-1} + F_{2k}) \\
&= F_{2k}^2 + \left(F_{2k}^2 - F_{2k-1}^2\right) + \left(F_{2k}^2 - F_{2k-1}F_{2k}\right) \\
&> F_{2k}^2.
\end{aligned}$$

Therefore,

$$\frac{1}{F_{4k-2}} - \frac{1}{F_{2k}^2} - \frac{1}{F_{4k+2}} < -\frac{1}{F_{4k-2}F_{4k+2}},$$

from which we arrive at

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_{2k}^2} &> \frac{1}{F_{4n-2}} - \frac{1}{F_{4mn+2}} + \frac{1}{F_{4n-2}F_{4n+2}} \\ &> \frac{1}{F_{4n-2}} + \frac{1}{F_{8n}} - \frac{1}{F_{4mn+2}} \\ &> \frac{1}{F_{4n-2}}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$\frac{1}{F_{4n-2}} < \sum_{k=n}^{mn}\frac{1}{F_{2k}^2} < \frac{1}{F_{4n-2}-1},$$

from which the desired result follows. □

As m tends to infinity in Theorem 3.4, we have the following consequence.

Corollary 3.5 *For all $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{F_{2k}^2}\right)^{-1} \right\rfloor = F_{4n-2} - 1. \tag{3.10}$$

Theorem 3.6 *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn}\frac{1}{F_{2k-1}^2}\right)^{-1} \right\rfloor = F_{4n-4}. \tag{3.11}$$

Proof It is obvious when $n = 1$, thus we assume that $n \geq 2$ in the following.
It follows from (2.3) that

$$\begin{aligned}F_{2k+1}^2 - F_{2k-1}^2 &= F_{4k}, \\ F_{2k-1}^2 - F_{2k-3}^2 &= F_{4k-4}.\end{aligned}$$

Therefore, applying (3.2), we deduce

$$\begin{aligned}F_{2k-1}^2(F_{4k} - F_{4k-4}) - F_{4k-4}F_{4k} &= F_{2k-1}^2\left(F_{2k+1}^2 - 2F_{2k-1}^2 + F_{2k-3}^2\right) \\ &\quad - \left(F_{2k+1}^2 - F_{2k-1}^2\right)\left(F_{2k-1}^2 - F_{2k-3}^2\right) \\ &= F_{2k-3}^2F_{2k+1}^2 - F_{2k-1}^4 \\ &= \left(F_{2k-3}F_{2k+1} - F_{2k-1}^2\right)\left(F_{2k-3}F_{2k+1} + F_{2k-1}^2\right) \\ &= F_{2k-3}F_{2k+1} + F_{2k-1}^2.\end{aligned} \tag{3.12}$$

For $k \geq 2$, we have

$$\begin{aligned}\frac{1}{F_{4k-4}} - \frac{1}{F_{2k-1}^2} - \frac{1}{F_{4k}} &= \frac{F_{2k-1}^2(F_{4k} - F_{4k-4}) - F_{4k-4}F_{4k}}{F_{4k-4}F_{2k-1}^2F_{4k}} \\ &= \frac{F_{2k-3}F_{2k+1} + F_{2k-1}^2}{F_{4k-4}F_{2k-1}^2F_{4k}} \\ &> 0,\end{aligned}$$

from which we derive

$$\sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} < \frac{1}{F_{4n-4}} - \frac{1}{F_{4mn}} < \frac{1}{F_{4n-4}}. \tag{3.13}$$

Employing (3.2) and (3.12), we obtain

$$F_{2k-1}^2(F_{4k} - F_{4k-4}) - (F_{4k-4} + 1)(F_{4k} + 1) = 2F_{2k-1}^2 + F_{2k-3}^2 - F_{2k+1}^2 < -F_{2k-1}^2,$$

where the last inequality follows from (3.3).

Now we see that, for $k \geq 2$,

$$\begin{aligned}\frac{1}{F_{4k-4}+1} - \frac{1}{F_{2k-1}^2} - \frac{1}{F_{4k}+1} &= \frac{F_{2k-1}^2(F_{4k} - F_{4k-4}) - (F_{4k-4}+1)(F_{4k}+1)}{(F_{4k-4}+1)F_{2k-1}^2(F_{4k}+1)} \\ &< \frac{-1}{(F_{4k-4}+1)(F_{4k}+1)},\end{aligned}$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} > \frac{1}{F_{4n-4}+1} - \frac{1}{F_{4mn}+1} + \frac{1}{(F_{4n-4}+1)(F_{4n}+1)}.$$

It is easy to see that

$$\begin{aligned}(F_{4n-4}+1)(F_{4n}+1) &= F_{4n-4}F_{4n} + F_{4n-4} + F_{4n} + 1 \\ &< F_{8n-4} + F_{8n-3} + F_{8n-1} + 1 \\ &= F_{8n} + 1.\end{aligned}$$

Hence,

$$\sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} > \frac{1}{F_{4n-4}+1} + \frac{1}{F_{8n}+1} - \frac{1}{F_{4mn}+1} \geq \frac{1}{F_{4n-4}+1}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\frac{1}{F_{4n-4}+1} < \sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} < \frac{1}{F_{4n-4}},$$

which completes the proof. □

Consequently, we have the following result.

Corollary 3.7 *If* $n \geq 1$, *then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{2k-1}^2} \right)^{-1} \right\rfloor = F_{4n-4}. \tag{3.15}$$

4 Conclusions

In this paper, we give the exact integral values of the reciprocal sums (resp., square sums) of the even and odd terms in the Fibonacci sequence. The results are new and important for those with closely related research interests. In addition, the methods used here are very elementary and can be extended to the investigation of other combinatorial sequences.

In a future paper, the reciprocal sums and the reciprocal square sums of the Fibonacci 3-subsequences will be presented.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to deriving all the results of this article, and read and approved the final manuscript.

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